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Scalar-gravitational perturbations and quasinormal modes in the five dimensional Schwarzschild black hole

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Abstract: We calculate the quasinormal modes (QNMs) for gravitational perturbations of the Schwarzschild black hole in the five dimensional (5D) spacetime with a continued fraction method. For all the types of perturbations (scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations), the QNMs associated with $l = 2$, $l = 3$, and $l = 4$ are calculated. Our numerical results are summarized as follows: (i) The three types of gravitational perturbations associated with the same angular quantum number l have a different set of the quasinormal (QN) frequencies; (ii) There is no purely imaginary frequency mode; (iii) The three types of gravitational perturbations have the same asymptotic behavior of the QNMs in the limit of the large imaginary frequencies, which are given by $\omega T_H^{-1} \rightarrow \log 3 + 2\pi i(n + 1/2)$ as $n \rightarrow \infty$, where ω, T_H, and n are the oscillation frequency, the Hawking temperature of the black hole, and the mode number, respectively.

Keywords: Quasinormal Modes, Gravitational Radiation, Higher Dimensions.

Contents

1. Introduction

The quasinormal (QN) ringing of a Schwarzschild spacetime was first observed in [1] through numerical calculations of the gravitational wave scattering by the black hole. Since then, the quasinormal modes (QNMs) of black holes have been extensively studied. The classical motivation behind the exploration of the QNMs of black holes is twofold: One is to answer the question of whether the spacetime is stable, and the other to know what kind of oscillations will be excited in the spacetime as some perturbations are given. The latter is quite important from the observational point of view because we could determine fundamental parameters of a black hole, such as the mass or the angular momentum, through the information of the QNMs. A number of studies on QNMs of several different spacetimes containing black holes have been done. (for a review see, e.g., [2, 3]). Recently QNMs have acquired a different status, since it was conjectured that they may be connected to black hole area quantization and quantum gravity [4, 5, 6].

Most studies on QNMs of black holes were restricted to the four dimensional (4D) case, compatible with astrophysical scenarios. However, motivated by the TeV-scale gravity proposals [7] for instance, higher-dimensional (higher-D) theories of gravity have recently attracted much attention. Up to recently, there were no master equations for examining the QNMs for gravitational perturbations of higher-D black holes. Due to the absence of these master equations, only the QNMs of the simplest test fields, namely massless scalar fields, around the higher-D black holes were calculated [8, 9, 10, 11].

The situation has changed recently. Kodama and Ishibashi [12] have derived master equations for gravitational perturbations in a higher-D Schwarzschild black hole spacetime. Putting $D = 2 + n$, it was shown that for the case $n \geq 3$, the gravitational perturbations can be divided into three classes, namely, scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations, according to their tensorial behavior on the n-sphere. The scalar-gravitational and vector-gravitational perturbations correspond to the polar and axial perturbations in the 4D spacetime. The tensor-gravitational perturbations are a new kind appearing in higher-Ds. Kodama and Ishibashi have also shown that all the types of gravitational perturbations can be reduced to a simple Schrödinger-type wave equation like Regge-Wheeler or Zerilli equations. It is important to ask the question of whether there is a special relationship among the scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations because in 4D the scalar-gravitational and vector-gravitational perturbations have a special relation yielding the same quasinormal (QN) frequencies. There seems to be no such relation between the scalar-gravitational and vector-gravitational perturbations [12].

Recently, some studies on the QNMs for gravitational perturbations of higher-D Schwarzschild black holes have appeared. Konoplya has calculated the fundamental QNMs of Kodama and Ishibashi's master equations with a JWKB approximation, and has found that three types of perturbations have different fundamental QN frequencies [13]. Berti, Cavaglià, and Gualtieri have also done similar calculations but for a wide range of angular eigenvalues l of perturbations [14]. Cardoso, Lemos, and Yoshida have calculated the QNMs for the vector-gravitational and tensor-gravitational perturbations up to higher-order modes [5]. As for the asymptotic behavior in the limit of highly damped modes, it has been shown that the three types of gravitational perturbations have the same behavior regardless of the spacetime dimensions or the angular quantum numbers of the perturbations [15, 5].

The purpose of this paper is to go a step further up and explore the QNMs in the special 5D higher-D black hole spacetime. In particular, we want to find out whether or not there is a relationship among the QNMs of the scalar-gravitational, vector-gravitational and tensor-gravitational perturbations in this 5D Schwarzschild black hole. The 5D case can be considered representative of the other higher-D cases. A prime purpose of the present study is to extensively calculate the QNMs for the scalar-gravitational

perturbations. The scalar-gravitational perturbations are the most important because they can be excited more easily than other types of perturbations. We calculate the QN frequencies not only for low-order modes but also for relatively higher-order modes, enabling us to discuss the asymptotic behavior of the highly damped QNMs. In the present study Nollert's approach is employed numerically to obtain the QN frequencies. The paper is organized as follows: In section 2 we present the basic equations employed for obtaining QNMs in the 5D Schwarzschild spacetime. Numerical results are given in section 3, and in section 4 we conclude.

2. Numerical Method

The general perturbation equations for the Schwarzschild black hole in $D = n + 2$ dimensions have been recently derived by Kodama and Ishibashi [12]. It has been shown that there are three completely decoupled classes in the perturbations for the $n \geq 3$ case, namely scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations, and that all types of perturbations can be cast into a Schrödinger-type wave equation, given by

$$f\frac{d}{dr}\left(f\frac{d\Phi}{dr}\right) + (\omega^2 - V(r))\Phi = 0\,, \tag{2.1}$$

where the function f is minus the g_{tt} component of the metric tensor in Schwarzschild coordinates, and given by

$$f = 1 - \frac{2M}{r^{n-1}} = 1 - x\,. \tag{2.2}$$

Here, r is the radial coordinate of the $n+2$ dimensional Schwarzschild spacetime, and ω is the oscillation frequency of the perturbations. M is a parameter related to the mass $\tilde{M}$ of the black hole via the relation

$$\tilde{M} = \frac{nMA_n}{8\pi G_{n+2}}\,, \tag{2.3}$$

where A_n is the area of a unit n-sphere, given by $A_n = 2\pi^{(n+1)/2}/\Gamma[(n+1)/2]$, and G_{n+2} stands for the $n+2$ dimensional Newton constant. The Hawking temperature of the black hole is $T_H = (n-1)/4\pi r_h$, where r_h is the horizon radius, defined by $r_h^{n-1} = 2M$.

In the master equation (2.1), depending on the type of perturbations, the effective potential can be written as

$$\frac{r^2V}{f} = \begin{cases} \frac{Q(r)}{16H(r)^2} & \text{for scalar} \\ \frac{(2l+n)(2l+n-2)}{4} - \frac{3n^2x}{2} & \text{for vector} \\ \frac{(2l+n)(2l+n-2)}{4} + \frac{n^2x}{2} & \text{for tensor}\,, \end{cases} \tag{2.4}$$

with

$$\begin{aligned} Q(r) &= n^4(n+1)^2x^3 + n(n+1)\{4(2n^2-3n+4)m + n(n-2)(n-4)(n+1)\}x^2 \\ &\quad - 12n\{(n-4)m + n(n+1)(n-2)\}mx + 16m^3 + 4n(n+2)m^2\,, \end{aligned} \tag{2.5}$$

$$H(r) = m + \frac{1}{2}n(n+1)x\,, \tag{2.6}$$

$$m = l(l+n-1) - n\,, \tag{2.7}$$

where l means the angular quantum number of perturbations, and $l \geq 2$ has been assumed. It is worthwhile to note that the scalar-gravitational and vector-gravitational perturbations are, respectively, higher-D

counterparts of the polar and axial perturbations in the 4D Schwarzschild black hole, while the tensor-gravitational perturbations are brand new in the sense that a counterpart of the tensor-gravitational perturbations does not appear in the 4D case. In the case of $n = 2$, the master equations for the scalar-gravitational and vector-gravitational perturbations therefore become the Zerilli equation and the Regge-Wheeler equation, respectively [17, 16]. Interestingly, the effective potential for the tensor-gravitational perturbations is exactly the same as that for the massless scalar field in the higher-D Schwarzschild black hole [8].

The QNMs of the higher-D Schwarzschild black hole are characterized by the boundary conditions of incoming waves at the black hole horizon and outgoing waves at spatial infinity, written as

$$\Phi(r) \to \begin{cases} e^{-i\omega r_*} & \text{as } r_* \to \infty \\ e^{i\omega r_*} & \text{as } r_* \to -\infty \,, \end{cases} \tag{2.8}$$

where the time dependence of perturbations has been assumed to be $e^{i\omega t}$. Here, r_* denotes the tortoise coordinate, defined by $dr_* = f^{-1}dr$. In order to obtain numerically the QN frequencies, we employ Nollert's method [18], an enhanced version of Leaver's continued fraction method [19], since it is nicely suitable to the study of the asymptotic behavior of QNMs in the limit of highly damped modes, the main concern in the present investigation.

In this study, we only consider the QNMs of the Schwarzschild black hole in the five dimensional spacetime, namely the $n = 3$ case. The tortoise coordinate is then reduced to

$$r_* = z^{-1} + \frac{1}{2z_1}\ln(z - z_1) - \frac{1}{2z_1}\ln(z + z_1)\,, \tag{2.9}$$

where $z = r^{-1}$ and $z_1 = r_h^{-1}$. The perturbation function Φ can be expanded around the horizon as

$$\Phi = e^{-i\omega z^{-1}}(z - z_1)^{\rho}(z + z_1)^{\rho}\sum_{k=0}^{\infty} a_k \left(\frac{z - z_1}{-z_1}\right)^k \,, \tag{2.10}$$

where $\rho = i\omega/2z_1$ and a_0 is taken to be $a_0 = 1$. Here, the expansion coefficients a_k for $k \geq 1$ are determined with the recurrence relation, the order of which is dependent on the functional form of the effective potential in equation (2.1). As shown by Cardoso, Lemos and Yoshida [5], the recurrence relation becomes a four-term relation for vector-gravitational and tensor-gravitational perturbations. In this paper we do not show an explicit expression of the four-term recurrence relation for the vector-gravitational and tensor-gravitational perturbations since it can be found in [5]. For the scalar-gravitational perturbations, on the other hand, the recurrence relation becomes eight-term relation, given by

$$\begin{aligned}
&c_0(0)a_1 + c_1(0)a_0 = 0\,,\\
&c_0(1)a_2 + c_1(1)a_1 + c_2(1)a_0 = 0\,,\\
&c_0(2)a_3 + c_1(2)a_2 + c_2(2)a_1 + c_3(2)a_0 = 0\,,\\
&c_0(3)a_4 + c_1(3)a_3 + c_2(3)a_2 + c_3(3)a_1 + c_4(3)a_0 = 0\,,\\
&c_0(4)a_5 + c_1(4)a_4 + c_2(4)a_3 + c_3(4)a_2 + c_4(4)a_1 + c_5(4)a_0 = 0\,,\\
&c_0(5)a_6 + c_1(5)a_5 + c_2(5)a_4 + c_3(5)a_3 + c_4(5)a_2 + c_5(5)a_1 + c_6(5)a_0 = 0\,,\\
&c_0(k)a_{k+1} + c_1(k)a_k + c_2(k)a_{k-1} + c_3(k)a_{k-2} + c_4(k)a_{k-3} + c_5(k)a_{k-4}\\
&+c_6(k)a_{k-5} + c_7(k)a_{k-6} = 0 \text{ , for } k = 6, 7 \cdots \,,
\end{aligned} \tag{2.11}$$

where

$$
\begin{aligned}
c_0(k) &= 2(1+k)(6+m)^2(1+k+2\rho)\,,\\
c_1(k) &= -(6+m)\{m^2+(78+5m)k^2+10m(1+2\rho)\rho+(30+5m+216\rho+20m\rho)k+12(1+5\rho+10\rho^2)\}\,,\\
c_2(k) &= [8(324+48m+m^2)k^2+m^2(1+32\rho^2)+12m(13-48\rho+112\rho^2)+36(41-128\rho+192\rho^2)\\
&\quad +16\{-198+540\rho+2m^2\rho+3m(-7+30\rho)\}k]/2\,,\\
c_3(k) &= [-4(1980+168m+m^2)(-2+k)^2-m^2(3+4\rho)^2-12m(69+192\rho+224\rho^2)\\
&\quad -36(129+520\rho+752\rho^2)-4(-2+k)\{m^2(3+4\rho)+96m(3+7\rho)+36(65+204\rho)\}]/4\,,\\
c_4(k) &= 12[6(25+m)k^2+m(25-44\rho+24\rho^2)+6(127-214\rho+96\rho^2)\\
&\quad +\{-660+588\rho+m(-22+24\rho)\}k]\,,\\
c_5(k) &= -3[4(81+m)k^2+m(29-40\rho+16\rho^2)+6(491-644\rho+216\rho^2)\\
&\quad +4\{-483+324\rho+m(-5+4\rho)\}k]\,,\\
c_6(k) &= 18\{227+16k^2-240\rho+64\rho^2+8(-15+8\rho)k\}\,,\\
c_7(k) &= -9(-9+2k+4\rho)^2\,.
\end{aligned}
\tag{2.12}
$$

It is understood that since the asymptotic form of the perturbations as $r_* \to \infty$ is written in terms of the variable z as

$$e^{-i\omega r_*} = e^{-i\omega z^{-1}}(z-z_1)^{-\rho}(z+z_1)^{\rho}\,, \tag{2.13}$$

the expanded perturbation function Φ defined by equation (2.10) automatically satisfy the QNM boundary conditions (2.8) if the power series converges for $0 \le z \le z_1$. After making a Gaussian elimination five times [20], we can reduce the eight-term recurrence relation to a three-term relation, which has the form

$$
\begin{aligned}
&\alpha_0 a_1 + \beta_0 a_0 = 0\,,\\
&\alpha_k a_{k+1} + \beta_k a_k + \gamma_k a_{k-1} = 0\,,\ \ k = 1, 2, \cdots\,.
\end{aligned}
\tag{2.14}
$$

Here, we omit the explicit expression for the final three-term recurrence relation because the numerical procedure to obtain the three-term recurrence relation is quite simple. Now that we have the three-term recurrence relation for determining the expansion coefficients a_k, according to Leaver [21, 19], the convergence condition for the expansion (2.10), namely the QNM conditions, can be written in terms of the continued fraction as

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\frac{\alpha_2\gamma_3}{\beta_3-}\cdots\,. \tag{2.15}$$

In order to use Nollert's method, with which relatively higher-order QNMs with large imaginary frequencies can be obtained, we have to know the asymptotic behavior of a_{k+1}/a_k in the limit of $k \to \infty$. Following Leaver [20], it is found that

$$\frac{a_{k+1}}{a_k} = 1 \pm 2\sqrt{\rho}\,k^{-1/2} + \left(2\rho - \frac{3}{4}\right)k^{-1} + \cdots\,, \tag{2.16}$$

Table 1: QN frequencies ωr_h of the scalar-gravitational perturbations for $l = 2$, $l = 3$ and $l = 4$.

n	$l = 2$	$l = 3$	$l = 4$
0	$0.9477 + 0.2561i$	$1.6056 + 0.3110i$	$2.1924 + 0.3293i$
1	$0.8512 + 0.8212i$	$1.5109 + 0.9528i$	$2.1149 + 0.9999i$
2	$0.6727 + 1.5431i$	$1.3325 + 1.6586i$	$1.9649 + 1.7078i$
3	$0.5121 + 2.4399i$	$1.1107 + 2.4693i$	$1.7593 + 2.4804i$
4	$0.4169 + 3.4140i$	$0.9091 + 3.3823i$	$1.5319 + 3.3352i$
5	$0.3619 + 4.4092i$	$0.7586 + 4.3533i$	$1.3215 + 4.2642i$
6	$0.3275 + 5.4102i$	$0.6524 + 5.3471i$	$1.1486 + 5.2403i$
7	$0.3043 + 6.4130i$	$0.5765 + 6.3485i$	$1.0130 + 6.2392i$
8	$0.2876 + 7.4164i$	$0.5204 + 7.3524i$	$0.9071 + 7.2474i$
9	$0.2751 + 8.4197i$	$0.4778 + 8.3569i$	$0.8232 + 8.2584i$
10	$0.2653 + 9.4229i$	$0.4445 + 9.3614i$	$0.7555 + 9.2698i$
11	$0.2574 + 10.426i$	$0.4178 + 10.366i$	$0.7000 + 10.281i$
12	$0.2510 + 11.429i$	$0.3961 + 11.370i$	$0.6537 + 11.290i$

where the sign for the second term in the right-hand side is chosen so as to be

$$\mathrm{Re}(\pm 2\sqrt{\rho}) < 0\,. \tag{2.17}$$

In actual numerical computations, it is convenient to solve the k-th inversion of the continue fraction equation (2.15), given by

$$\begin{aligned}&\beta_k - \frac{\alpha_{k-1}\gamma_k}{\beta_{k-1}-}\frac{\alpha_{k-2}\gamma_{k-1}}{\beta_{k-2}-}\cdots\frac{\alpha_0\gamma_1}{\beta_0}\\ &= \frac{\alpha_k\gamma_{k+1}}{\beta_{k+1}-}\frac{\alpha_{k+1}\gamma_{k+2}}{\beta_{k+2}-}\cdots\,.\end{aligned} \tag{2.18}$$

The asymptotic form (2.16) plays an important role in Nollert's method when the infinite continued fraction in the right-hand side of equation (2.18) is evaluated [18].

3. Numerical Results

In order to confirm that our approach to obtain the QNMs for the scalar-gravitational perturbations works well, we first calculate the QNMs in the 4D case, namely the QNMs of Zerilli equation. For the Zerilli equation, eigenfunction is expanded around the horizon as

$$\Phi = e^{-i\omega z^{-1}}(z - z_1)^{i\omega z_1^{-1}} z^{i\omega z_1^{-1}} \sum_{k=0}^{\infty} a_k \left(\frac{z - z_1}{-z_1}\right)^k \,, \tag{3.1}$$

and the corresponding recurrence relation becomes a five-term relation. The numerical result obtained are in good agreement with the QN frequencies of the Regge-Wheeler equations, which are exactly the same as those of the Zerilli equation, showing that our numerical approach works quite well.

We calculate the QNMs associated with $l = 2, 3, 4$ for the scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations in the five dimensional Schwarzschild black with Nollert's numerical approach [18]. In order to discuss the asymptotic behavior of the QNMs in the limit of large imaginary

Table 2: QN frequencies ωr_h of the vector-gravitational perturbations for $l=2$, $l=3$ and $l=4$.

n	$l=2$	$l=3$	$l=4$
0	$1.1340 + 0.3275i$	$1.7254 + 0.3338i$	$2.2805 + 0.3400i$
1	$0.9474 + 1.0220i$	$1.6181 + 1.0199i$	$2.2003 + 1.0317i$
2	$0.5429 + 1.9247i$	$1.4136 + 1.7655i$	$2.0452 + 1.7598i$
3	$0.4357 + 3.1209i$	$1.1485 + 2.6098i$	$1.8314 + 2.5508i$
4	$0.3996 + 4.1798i$	$0.8798 + 3.5543i$	$1.5914 + 3.4220i$
5	$0.3996 + 4.1798i$	$0.6275 + 4.5609i$	$1.3630 + 4.3662i$
6	$0.3715 + 5.2168i$	$0.3095 + 5.6127i$	$1.1668 + 5.3579i$
7	$0.3502 + 6.2441i$	$0.3160 + 7.0116i$	$1.0034 + 6.3729i$
8	$0.3335 + 7.2654i$	$0.3597 + 8.0670i$	$0.8651 + 7.3974i$
9	$0.3202 + 8.2827i$	$0.3696 + 9.1021i$	$0.7430 + 8.4249i$
10	$0.3093 + 9.2970i$	$0.3695 + 10.129i$	$0.6290 + 9.4526i$
11	$0.3002 + 10.309i$	$0.3657 + 11.151i$	$0.5140 + 10.480i$
12	$0.2925 + 11.320i$	$0.3604 + 12.169i$	$0.3821 + 11.508i$

Table 3: QN frequencies ωr_h of the tensor-gravitational perturbations for $l=2$, $l=3$ and $l=4$.

n	$l=2$	$l=3$	$l=4$
0	$1.5106 + 0.3575i$	$2.0079 + 0.3558i$	$2.5063 + 0.3550i$
1	$1.3927 + 1.1046i$	$1.9170 + 1.0853i$	$2.4327 + 1.0764i$
2	$1.1938 + 1.9457i$	$1.7483 + 1.8697i$	$2.2915 + 1.8328i$
3	$0.9944 + 2.8990i$	$1.5380 + 2.7393i$	$2.0991 + 2.6477i$
4	$0.8460 + 3.9147i$	$1.3365 + 3.6931i$	$1.8852 + 3.5353i$
5	$0.7436 + 4.9483i$	$1.1737 + 4.6991i$	$1.6828 + 4.4894i$
6	$0.6711 + 5.9832i$	$1.0503 + 5.7271i$	$1.5112 + 5.4882i$
7	$0.6174 + 7.0150i$	$0.9565 + 6.7613i$	$1.3730 + 6.5100i$
8	$0.5760 + 8.0431i$	$0.8836 + 7.7954i$	$1.2627 + 7.5414i$
9	$0.5429 + 9.0678i$	$0.8254 + 8.8275i$	$1.1736 + 8.5755i$
10	$0.5158 + 10.090i$	$0.7778 + 9.8568i$	$1.1006 + 9.6093i$
11	$0.4931 + 11.109i$	$0.7380 + 10.883i$	$1.0397 + 10.641i$
12	$0.4738 + 12.126i$	$0.7042 + 11.908i$	$0.9880 + 11.671i$

frequencies, the modes are obtained up to relatively higher-order modes, associated with the mode number of $n \sim 300$. Note that the results for the scalar-gravitational perturbations are newly obtained results, while the results for the vector-gravitational and tensor-gravitational perturbations have been already calculated in [5]. In the present calculations we have found no unstable mode whose frequency has a negative imaginary part. This is consistent with the results by Kodama and Ishibashi [22], who proved all the higher-D Schwarzschild black holes to be stable against small non-radial disturbance.

In Tables 1 through 3, the oscillation frequencies of the thirteen low-lying QNMs associated with $l = 2, 3, 4$ for the scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations are, respectively, tabulated. In the tables, the frequencies are shown in the units of the inverse of the horizon radius of the black hole because the horizon radius gives us one of the most natural units for the low-order QN frequencies. Note that another convenient frequency unit, suitable to the asymptotic behavior of highly damped QNMs, is the Hawking temperature of the black hole, given by $T_H = (n-1)/4\pi r_h$ (see,

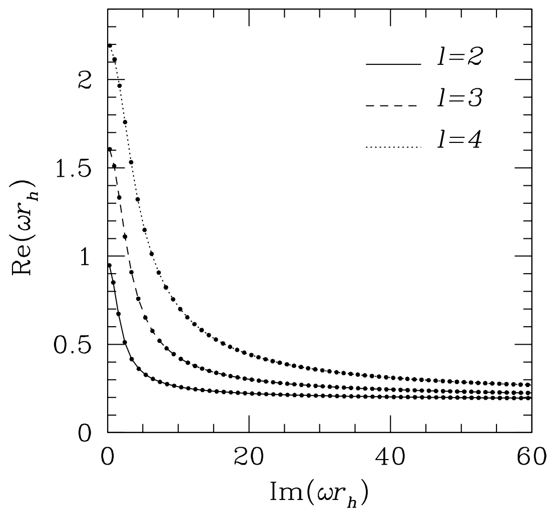

Figure 1: Real parts of the non-dimensional QNM frequencies, ωr_h, given as a function of the imaginary parts of the frequencies for the gravitational scalar-gravitational perturbations associated with $l = 2$, $l = 3$, and $l = 4$. First sixty QNMs are displayed. The solid circles are used to indicate the QN frequencies.

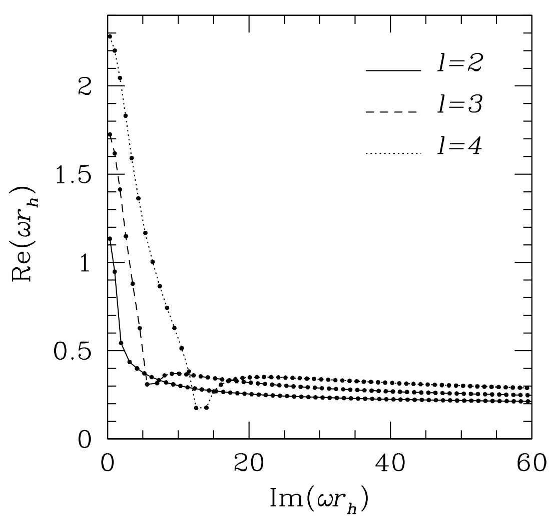

Figure 2: Same as Fig. 1 but for vector-gravitational perturbations.

e.g., [11, 15, 5]).

In order to check our numerical code, first of all, let us compare our results with those obtained by Konoplya [13], who calculated fundamental QNMs in the higher-D Schwarzschild black hole with the sixth-order JWKB approximations. Konoplya's frequency of the QNMs associated with $l = 3$ for the

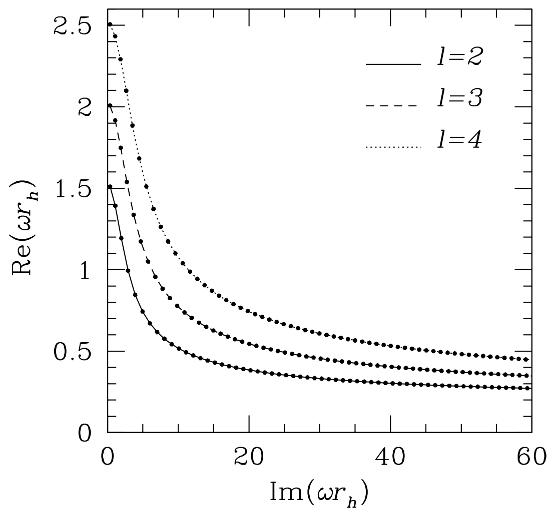

Figure 3: Same as Fig. 2 but for the tensor-gravitational perturbations.

scalar-gravitational perturbations in the five-dimensional case is given by $\omega r_h = 1.6044 + 0.3127i$ in the units we use, which is the only available data for the comparison. It is then found that our result is in good agreement with that of Konoplya, and that the relative difference between two frequencies is less than 1%.

In the 4D Schwarzschild spacetime, there are two important properties of the QN frequencies: (i) Two types of gravitational perturbations, namely axial and polar perturbations, associated with the same angular eigenvalue l possess exactly the same set of QN frequencies; (ii) There is a purely imaginary frequency. Interestingly, these two properties are attributed to the existence of a special relation between effective potentials of corresponding wave equations for scalar-gravitational and vector-gravitational perturbations. If we write two potentials for scalar-gravitational and vector-gravitational perturbations as V_+ and V_-, respectively, the existence of this special relation is equivalent to the existence of a function F satisfying

$$V_\pm = \pm f \frac{dF}{dr} + F^2 + cF \,, \tag{3.2}$$

where c is some constant [23]. For the higher-D case, Kodama and Ishibashi found that the function F satisfying equation (3.2) for scalar- and vector-gravitational perturbations does not exist in the $D > 4$ spacetime [12]. In the five dimensional case, due to the conclusion of Kodama and Ishibashi, it is expected that the three decoupled types of gravitational perturbations have different sets of the QN frequencies, and that there is no purely imaginary frequency of the QNMs. Our numerical results are consistent with these predictions. On Tables 1 through 3, it is observed that the three types of gravitational perturbations do not have the same set of QN frequencies even when the modes are associated with the same angular quantum number l. We also found no purely imaginary frequency of the QNMs.

In Figs. 1 through 3, we also display the real parts of the frequencies, Re(ωr_h), versus the the imaginary parts of the frequencies, Im(ωr_h), for the oscillation frequencies of the lower-lying QNMs associated with $l = 2, 3, 4$ for the scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations, respectively. In these figures, the first sixty QNM frequencies are exhibited. In Figs. 1-3, we can observe that the

real parts of the QNMs frequencies are monotonically decreasing functions of the imaginary parts of the frequencies except for the cases of the vector-gravitational perturbations associated with $l = 3$ and $l = 4$. The behavior of the QNMs of the vector-gravitational perturbations associated with $l = 3$ and $l = 4$ are similar to those of the gravitational perturbations in the 4D Schwarzschild black hole: the real parts of frequencies first decrease to a minimum value, increase to the local maximum, and then approach the asymptotic constant values, as the imaginary parts of the frequencies are increased.

Finally, let us discuss the asymptotic behavior of the QNMs in the limit of highly damped modes. In the Schwarzschild black hole in the $D = n + 2$ dimensional spacetime, the asymptotic QN frequencies in the limit of highly damped modes are given by

$$\omega T_H^{-1} = \log 3 + 2\pi i(n + \frac{1}{2}) , \tag{3.3}$$

regardless of the spacetime dimensions and the angular quantum number l [11, 15, 5]. On the other hand, higher-order corrections of the frequency (3.3) depend both on the spacetime dimensions and the angular quantum number l [5]. For the vector-gravitational and tensor-gravitational perturbations, the asymptotic behavior were already investigated in detail by Cardoso, Lemos and Yoshida [5]. Thus, we only consider the case of the scalar-gravitational perturbations, here. As shown in equation (3.3), we have $\mathrm{Re}(\omega r_h) \to (\log 3)/2\pi$ as $\mathrm{Im}(\omega r_h) \to \infty$ for the 5D case. Fig. 1 shows that the real parts of the QN frequencies approach to the asymptotic constant value $(\log 3)/2\pi$ as the imaginary parts of the frequencies increase, and that our numerical results are consistent with the analytical prediction of the asymptotic behavior in the limit of large imaginary frequency. Unfortunately, we cannot however obtain the QNMs associated with sufficiently large mode number with our numerical code. The reason of this is unclear, but we may guess that this is because the recurrence relation (2.11) for obtaining the QN frequencies becomes eight-term relation in the case of the scalar-gravitational perturbations. We therefore have failed to extract the detailed information of the asymptotic behavior of the QNM in the scalar-gravitational perturbations.

4. Conclusion

We have calculated the QNMs for the three types of gravitational perturbations of the 5D Schwarzschild black hole through a continued fraction method. In order to examine the QNMs, we made use of Schrödinger-type wave equations for determining the dynamics of the gravitational perturbations [12]. To apply the continued fraction method, we expanded the eigenfunctions around the black hole horizon in terms of a Fröbenius series. It was found that scalar-gravitational perturbations obey an eight-term recurrence relation, and vector-gravitational and tensor-gravitational perturbations obey a four-term recurrence relation. For all the types of perturbations, the QNMs associated with $l = 2$, $l = 3$, and $l = 4$ were calculated in the present study. Our numerical results are summarized as follows; (i) The three types of gravitational perturbations have different QNM frequencies; (ii) There is no purely imaginary frequency mode (the above two results are consistent with Kodama and Ishibashi's results that there is no function satisfying equation (3.2)); (iii) The QNMs belonging to the three types of gravitational perturbations have the same asymptotic behavior in the limit of the large imaginary frequencies, given by $\omega T_H^{-1} \to \log 3 + 2\pi i(n + 1/2)$ as $n \to \infty$; This asymptotic behavior was already expected [11, 15].

For the black hole dynamics, the lower-order QNMs play an important role after the initial non-linear effects have become insignificant. In the 4D Schwarzschild black hole, one can guess which frequency will be excited in the ringing phase since the axial and polar QNMs have the same frequencies. On the other hand, there are three different sets of QNM frequencies for the Schwarzschild black hole in

$D \geq 5$ dimensions. Our results show that the fundamental QNM for the scalar-gravitational perturbations have the smallest frequency and the longest damping-time. This means that the QNMs for the scalar-gravitational perturbations can live longer than other types of perturbations if all the types of perturbations are excited with the same initial amplitudes. Probably, in general situations things are not so simple. What type of perturbations will be excited will be strongly dependent on the situation. In order to expect what frequency will be excited in the ringing phase, one has to model the source terms of the master equations. So far, only the case where a test particle radially falls into a black hole has been considered [14]. In this case, due to the symmetry of the motion, only the scalar-gravitational perturbations are excited. The general case remains as a work to be solved.

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